

# HARDY-LITTLEWOOD INEQUALITIES FOR RIESZ'S POTENTIAL: low bounds estimations for different powers.

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Abstract.

*In this article we obtain the non - asymptotical low estimations for bilinear Riesz's functional through the Lebesgue spaces norms by means of building of some examples.*

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## 1. INTRODUCTION. STATEMENT OF PROBLEM.

The linear integral operator  $I_\alpha f(x)$ , or, more precisely, the *family of operators* of a view

$$u(x) = I_\alpha f(x) = \int_{R^d} \frac{f(y) dy}{|x - y|^{d-\alpha}}$$

is called *Riesz's integral operator*, or simply *Riesz's potential*, or *fractional integral*.

The bilinear *Riesz's functional*  $B_{\alpha,d}(f, g) = B_\alpha(f, g) = B(f, g)$  may be defined as follows:

$$B(f, g) = \int_{R^d} \int_{R^d} \frac{f(y) g(x) dx dy}{|x - y|^{d-\alpha}}.$$

It is evident that

$$B_{\alpha,d}(f, g) = (I_\alpha f, g),$$

where  $(f, g)$  denotes ordinary inner product of a (measurable) functions  $f$  and  $g$ :

$$(f, g) = \int_{R^d} f(x) g(x) dx,$$

which is defined, e.g. when  $f \in L_p$ ,  $g \in L_q$ ,  $p, q > 1$ ,  $1/p + 1/q = 1$ ,  $L_p = L_p(R^d)$  is the classical Lebesgue space of all the measurable functions  $f : R^d \rightarrow R$  with finite norm

$$|f|_p \stackrel{\text{def}}{=} \left( \int_{R^d} |f(x)|^p dx \right)^{1/p}; \quad f \in L_p \Leftrightarrow |f|_p < \infty.$$

Obviously,

$$\sup_{|g|_q=1} |B_{\alpha,d}(f, g)| = |I_{\alpha}f|_p.$$

Here  $\alpha = \text{const} \in (0, d)$  and  $|y|$  denotes usually Euclidean norm of the finite dimensional vector  $y$ . In particular, if  $y \in R^1$ ,  $|y|$  denotes ordinary absolute value of the number  $y$ .

In the case  $d \geq 2$  and  $\alpha = 2$   $u(x)$  coincides with the classical Newton's potential.

The operators  $I_{\alpha}$  and functionals  $B_{\alpha}$  are used in the theory of Fourier transform, theory of Partial Differential Equations, probability theory (study of potential functions for Markovian processes and spectral densities for stationary random fields), in the functional analysis, in particular, in the theory of interpolation of operators etc., see for instance [2], [16], [7], [9], [15], [12] etc.

We denote also  $L(a, b) = \cap_{p \in (a, b)} L_p$ .

**We will investigate the estimations of a view:**

$$|B_{\alpha,d}(f, g)| \leq K_{\alpha,d}(r, s) |f|_r |g|_s. \quad (1)$$

It is known [8], [10], [21] that the inequality (1) is possible only in the case when

$$r, s > 1, \frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{d}. \quad (2)$$

We will denote the set of all such a values  $r, s, r, s > 1$  as  $G = G_{\alpha,d}$ .

*We will suppose further that the condition (2) is satisfied:  $(r, s) \in G_{\alpha,d}$ .*

The exact value of the constant  $K_{\alpha,d}(r, s) = K(r, s)$ , i.e. the value

$$V_{\alpha,d}(r, s) = V(r, s) = \sup_{f \in L_r, f \neq 0} \sup_{g \in L_s, g \neq 0} \frac{B_{\alpha,d}(f, g)}{|f|_r |g|_s} \quad (3)$$

is now known only in the case  $r = s = 2d/(d + \alpha)$ . Namely, it is proved in the articles [8], [10] (in our notations) that

$$V_{\alpha,d}(2d/(d + \alpha), 2d/(d + \alpha)) = \pi^{0.5(d-\alpha)} \frac{\Gamma(\alpha/2)}{\Gamma(0.5(d + \alpha))} \left[ \frac{\Gamma(d)}{\Gamma(d/2)} \right]^{\alpha/d}.$$

In the book [7], p.98 is obtained the following *upper* estimate for the value  $V_{\alpha,d}(r, s)$ :

$$V_{\alpha,d}(r, s) \leq (rs)^{-1} \cdot \alpha^{-1} \cdot (\omega(d-1))^{1-\alpha/d} \cdot d^{\alpha/d} \cdot \left[ \frac{r^{1-\alpha/d}}{(r-1)^{1-\alpha/d}} + \frac{s^{1-\alpha/d}}{(s-1)^{1-\alpha/d}} \right]. \quad (4)$$

Note that the right-hand side of inequality (4) allows a simple estimation:

$$V_{\alpha,d}(r, s) \leq C_{1,d} \alpha^{-1} [(r-1)(s-1)]^{\alpha/d-1}. \quad (4.a)$$

WE CAN NOT CALCULATE THE EXACT VALUES OF  $V(r, s)$  FOR THE DIFFERENT VALUES  $r, s : r \neq s$ , BUT WE CAN FIND THE ASYMPTOTICALLY EQUIVALENT TO THE RIGHT-HAND SIDE AS  $r \rightarrow 1 + 0$  AND  $s \rightarrow 1 + 0$  OF THE INEQUALITY (4.A) LOW BOUND FOR THE VALUE  $V_{\alpha,d}(r, s)$  BIDE-SIDE ESTIMATIONS FOR THESE VALUES OF A VIEW:

$$V_{\alpha,d}(r, s) \geq \frac{K(d) \alpha^{-1}}{[(r-1)(s-1)]^{1-\alpha/d}}; \quad (5)$$

here  $(r, s) \in G_{\alpha,d}$ ,  $K(d)$  is finite positive function on the variable  $d$ .

Evidently, the estimations (4), (4a) and (5) may be rewritten as follows:

$$\frac{K_3(d) \alpha^{-1}}{[(r-1)(s-1)]^{1-\alpha/d}} \leq V_{\alpha,d}(r,s) \leq \frac{K_4(d) \alpha^{-1}}{[(r-1)(s-1)]^{1-\alpha/d}}, \quad (5.a)$$

$\forall d = 1, 2, \dots$   $0 < K_3(d) \leq K_4(d) < \infty$ .

We dare formulate as a hypotheses the following equality:

$$\sup_{r,s \in G_{\alpha,d}} \{ [(r-1)^{1-\alpha/d} (s-1)^{1-\alpha/d}] V_{\alpha,d}(r,s) \} = K_5(\alpha, d) \alpha^{-1},$$

where the powers  $1 - \alpha/d$ ,  $1 - \alpha/d$  are exact and the function  $K_5(\alpha, d)$  is positive and continuous in the closed interval  $\alpha \in [0, d]$ .

The article is organized as follows. In the next section we obtain the main result: low bounds for Riesz's linear operator and correspondent bilinear form in the Lebesgue spaces.

In the third section we consider the case when instead the whole space  $R^d$  in the integral in (1) is some bounded domain.

The last section contains some slight generalizations of obtained results.

We use symbols  $C(X, Y)$ ,  $C(p, q; \psi)$ , etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like  $C_1(X, Y)$  and  $C_2(X, Y)$ . The relation  $g(\cdot) \asymp h(\cdot)$ ,  $p \in (A, B)$ , where  $g = g(p)$ ,  $h = h(p)$ ,  $g, h : (A, B) \rightarrow R_+$ , denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$

The symbol  $\sim$  will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

$$I(x \in A) = 1, x \in A, I(x \in A) = 0, x \notin A;$$

here  $A$  is a measurable set.

Other notations. We denote as usually

$$\Omega(d) = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}, \quad \omega(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad \bar{\omega}(d) = \max(1, \omega(d)).$$

The value  $\Omega(d)$  is the volume of the unit ball in the space  $R^d$ :

$$m(B(x, r)) = \Omega(d) r^d; \quad B(x, r) = \{y, y \in R^d, |x - y| \leq r\},$$

$m(\cdot)$  denotes ordinary Lebesgue measure and  $\Gamma(\cdot)$  denotes usually Gamma - function.

The value  $\omega(d)$  is the area of the unit sphere in this space.

Note that  $\omega(d) = d \Omega(d)$ .

Further, we need to use the so-called maximal operator  $Mf(x)$ :

$$Mf(x) \stackrel{\text{def}}{=} \sup_{r>0} \left[ \Omega(d)^{-1} r^{-d} \int_{y:|y-x|\leq r} |f(y)| dy \right].$$

It is known ([16], p. 173 - 188) that

$$|Mf|_p \leq C(d) \frac{p}{p-1} |f|_p, \quad p \in (1, \infty), \quad C(d) \in (0, \infty).$$

The minimal value of the constant  $C(d)$  from the last inequality will be denoted by  $S(d)$ , ("Stein's constant"), on the other hand:

$$S(d) \stackrel{\text{def}}{=} \sup_{p \in (1, \infty)} \sup_{f \neq 0, f \in L(1, \infty)} \frac{|Mf|_p}{p |f|_p / (p - 1)}.$$

It is evident that

$$\inf_{p \in (1, \infty)} S(d) \geq 1, \quad d = 1, 2, \dots$$

The first upper estimation for the value  $S(d)$  was obtained in the classical book of E.M.Stein ([16], p. 173 - 188):

$$S(d) \leq 2 \cdot 5^d.$$

In the article [4] it is proved that  $S(2) \leq 2$ . In the next works of E.M.Stein [16], [17], [18],[19],it was obtained the following estimations for  $S(d)$  :

$$S(d) \leq C_1 \sqrt{d}, \quad S(d) \leq C_2$$

with *some absolute* constants  $C_1$  and  $C_2$ .

All the passing to the limit in this article may be grounded by means of Lebesgue dominated convergence theorem.

### Some comments about upper estimations for Riesz potential.

The original proof of the inequality (4) in the book [7], pp. 98-112 based on the *weak* Young's inequality

$$\int_{R^d} \int_{R^d} f(x)g(x-y)h(y) dx dy \leq C(p, q, r) |f|_p |g|_{q,w} |h|_r,$$

where  $p, q, r > 1, 1/p + 1/q + 1/r = 2$  and  $|g|_{q,w}$  denotes the weak Lebesgue norm of order  $q$ , which may be defined (up to norm equivalence) as

$$|g|_{q,w} = \sup_{A, m(A) \in (0, \infty)} (m(A))^{1/q-1} \int_A |g(x)| dx,$$

where  $m(A)$  denotes usually Lebesgue measure of (measurable) set  $A$ , in contradiction to the classical Young's inequality

$$\int_{R^d} \int_{R^d} f(x)g(x-y)h(y) dx dy \leq |f|_p |g|_q |h|_r,$$

where again  $p, q, r > 1, 1/p + 1/q + 1/r = 2$ .

*But we will offer an another proof, which gives the possibility for many generalizations.*

We will consider in this subsections only the values  $p$  from the open interval  $p \in (1, d/\alpha)$  and denote  $q = q(p) = pd/(d - \alpha p)$ ; or equally

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d};$$

evidently,  $q \in (d/(d - \alpha), \infty)$ .

The inverse function to the function  $q = q(p)$  has a view  $p = p(q) = dq/(d + \alpha q)$ ; note that if  $p \rightarrow 1 + 0 \Rightarrow q \rightarrow d/(d - \alpha) + 0$  and if  $p \rightarrow d/\alpha - 0 \Rightarrow q \rightarrow \infty$ .

Note that the case  $p = 1$  and  $p = d/\alpha$ , i.e. when  $f \in L_1(R^d)$  or  $f \in L_{d/\alpha}(R^d)$  is considered, e.g., in [1], p. 56.

More detail,

$$p - 1 = \frac{(d - \alpha)(q - d/(d - \alpha))}{d + \alpha q}$$

and

$$\frac{d}{\alpha} - p = \frac{d^2}{\alpha(d + \alpha q)}.$$

**Theorem 1.**

$$\begin{aligned} |I_\alpha f|_q &\leq \left[ S(d) \overline{\omega(d)} \right] \cdot |f|_p \cdot p^{(\alpha p - d)(p-1)} \cdot (p-1)^{\alpha(p-1)/d} \times \\ &[(p-1)(d/\alpha - p)]^{\alpha/d-1} \cdot \left[ 1 + \frac{(p-1)^{1-1/p}}{\alpha p} (d - \alpha p) \right] (\stackrel{def}{=} R), \end{aligned} \quad (6)$$

which is equivalent the inequality (4).

**Remark 1.** Note that the right hand side of the inequality (6), say  $R$ , may be estimated as follows:

$$R \leq \left[ S(d) \overline{\omega(d)} \right] \cdot |f|_p \cdot \frac{2d^2/\alpha}{[(p-1)(d/\alpha - p)]^{1-\alpha/d}}. \quad (7)$$

**Remark 2.** The last estimation (7) improved the main result of the article [14].

**Proof** of the theorem 1 (briefly).

In the book [1], pp. 49 - 54 is described a very interest approach, which we will use here.

Let  $\chi = \chi(z)$  be a positive at  $z \in (0, \infty)$  continuous decreasing function such that  $\phi(\infty) \stackrel{def}{=} \lim_{z \rightarrow \infty} \phi(z) = 0$ . We define also a function

$$\Phi(z) = \int_z^\infty \chi(t) dt,$$

if there exists.

We have for the values  $\delta \in (0, \infty)$  analogously to the assertion in [1], p. 49 - 51:

$$\int_{y:|x-y|<\delta} \Phi(|x-y|) f(y) dy = \int_0^\delta \chi(r) \int_{y:|x-y|<r} f(y) dy + \Phi(\delta) \int_{y:|x-y|<\delta} f(y) dy.$$

Without loss of generality we can assume that the function  $f(\cdot)$  is non - negative.

As long as

$$\begin{aligned} \int_{y:|x-y|\leq r} f(y) dy &\leq \Omega(d) r^d Mf(x), \\ \int_{y:|x-y|\leq \delta} f(y) dy &\leq \Omega(d) \delta^d Mf(x), \end{aligned}$$

we obtain the estimate

$$\int_{y:|x-y|<\delta} \Phi(|x-y|) f(y) dy \leq \Omega(d) Mf(x) \cdot \left[ \int_0^\delta r^d \chi(r) dr + \delta^d \Phi(\delta) \right]$$

$$\stackrel{def}{=} Mf(x) A_{\chi,d}(\delta).$$

Further, we have denoting  $s = p/(p-1)$  and using Hölder's inequality:

$$\begin{aligned} \int_{y:|x-y|\geq\delta} \Phi(|x-y|) f(y) dy &\leq |f|_p \cdot \left[ \int_{y:|x-y|>\delta} \Phi^s(|x-y|) dy \right]^{1/s} = \\ &\omega(d) |f|_p \left[ \int_{\delta}^{\infty} r^{d-1} \Phi^s(r) dr \right]^{1/s} = D_{\chi}(p, \delta) |f|_p, \end{aligned}$$

where

$$D_{\chi}(p, \delta) \stackrel{def}{=} \omega(d) \left[ \int_{\delta}^{\infty} r^{d-1} \Phi^s(r) dr \right]^{1/s},$$

if there exists for some values  $s$  from some non-trivial interval  $s \in (d/(d-\alpha), s_0)$ ; if  $s_0 < \infty$ , then we define formally  $D_{\chi}(p, \delta) = +\infty$ .

We conclude taking into account the partition

$$\begin{aligned} \int_{R^d} \Phi(|x-y|) f(y) dy &= \int_{y:|x-y|<\delta} \Phi(|x-y|) f(y) dy \\ &+ \int_{y:|x-y|\geq\delta} \Phi(|x-y|) f(y) dy : \\ \int_{R^d} \Phi(|x-y|) f(y) dy &\leq Mf(x) A_{\chi,d}(\delta) + D_{\chi}(p, \delta) |f|_p. \end{aligned}$$

Therefore, we can make the optimization over  $\delta$ ,  $\delta \in (0, \infty)$ :

$$\begin{aligned} \int_{R^d} \Phi(|x-y|) f(y) dy &\leq \inf_{\delta>0} [Mf(x) A_{\chi,d}(\delta) + D_{\chi}(p, \delta) |f|_p] \\ &\stackrel{def}{=} H(p, Mf(x), |f|_p). \end{aligned}$$

Solving the last inequality, we obtain denoting

$$w(x) = \int_{R^d} \Phi(|x-y|) f(y) dy :$$

the inequality of a view

$$G(p, w(x), |f|_p) \leq (Mf(x))^p,$$

and after the integration with the corresponding power

$$\begin{aligned} \int_{R^d} G(p, w(x), |f|_p) dx &\leq |Mf(x)|_p^p \leq C^p(\alpha, d) |f|_p^p (p-1)^{-p}, \\ \left[ \int_{R^d} G(p, w(x), |f|_p) dx \right]^{1/p} &\leq |Mf(x)|_p \leq C(\alpha, d, \chi) |f|_p (p-1). \end{aligned}$$

Since the relation between the functions  $f(\cdot)$  and  $w(\cdot)$  is linear, the last inequality has a view

$$|w|_q \leq C_2(\alpha, p, \chi(\cdot)) |f|_p.$$

Choosing the function  $\Phi(r)$  as follows:

$$\Phi(r) = r^{\alpha-d}$$

and taking into account the Stein's estimations for the maximal functions  $Mf(x)$ , we obtain the assertion of theorem 1 after complicate calculations.

□

## 2. MAIN RESULT: LOW BOUNDS FOR RIESZ'S POTENTIAL.

In this section we built some examples in order to illustrate the exactness of upper estimations (4).

We consider here only more hard case  $d \geq 2$ ; the one-dimensional case  $d = 1$  is considered in the work [14].

Before the formulating of the main result of this section, we must introduce several new notations.

$$a = a(\alpha, d) := e^{1/e} \cdot \max \left[ \frac{\omega(d)}{\alpha}, \left( \frac{\omega(d)}{\alpha} \right)^{d/\alpha} \right],$$

$$m = m(\alpha, d) := \min (1, (\omega(d)/d)^{1-\alpha/d}),$$

$$A = A(\alpha, d) := 4\pi/(9\alpha) \cdot \omega(d-1) \cdot 2^{-d} \cdot m(\alpha, d),$$

$$n = n(\alpha, d) := \max (\omega(d)/d, (\omega(d)/d)^{\alpha/d}),$$

$$D = D(\alpha, d) := 3^{-1} \cdot 4 \cdot 5^{\alpha-d} \cdot \omega(d) \cdot \min(1, (\omega(d))^{d/(d-\alpha)} \cdot (d^2/\alpha)^{-2-\alpha/d}).$$

Let us introduce the following important function:  $F(p) = F_{\alpha,d}(p) :=$

$$0.5 \frac{A \cdot (p-1)^{1/p+(d-\alpha)/\alpha} + D \cdot (d/\alpha - p)^{1/p+(2d-\alpha)/d}}{a \cdot (p-1)^{1/p} + n \cdot (d/\alpha - p)^{\alpha/d}}. \quad (8)$$

Since the function  $p \rightarrow F(p)$  is continuous in the *closed* interval  $p \in [1, d/\alpha]$  and is positive, we can introduce the strong positive variable

$$R(\alpha, d) := \inf_{p \in [1, d/\alpha]} F_{\alpha,d}(p). \quad (9)$$

**Theorem 2.** For  $(r, s) \in G_{\alpha,d}$  there holds:

$$V_{\alpha,d}(r, s) \geq \frac{R(\alpha, d)}{[(r-1)(s-1)]^{1-\alpha/d}}, \quad (10)$$

**Remark 3.** Note that the  $R(\alpha, d)$  may be estimated from below as follows:  $R(\alpha, d) \geq K_6(d) \alpha^{-1}$ ,  $\alpha \in (0, 1)$ , which asymptotically coincides with the upper bound given by theorem 1.

**Proof.** We will consider two examples of a functions from the set  $L(1, d/\alpha)$ .

**First example.**

$$f_0(x) = |x|^{-d} I(|x| > 1).$$

We find by direct calculations using multidimensional polar coordinates:

$$|f_0|_p = (\omega(d))^{1/p} d^{-1/p} (p-1)^{-1/p} \leq n(\alpha, d) (p-1)^{-1/p}, p > 1;$$

$$u_0(x) := I_\alpha f_0(x) \geq C_\alpha(d) |x|^{\alpha-d} |\log|x|| I(|x| > 1),$$

where

$$C_\alpha(d) = 4 \cdot 5^{\alpha-d} \cdot \omega(d);$$

$$|u_0|_q \geq C_\alpha(d) \cdot (\omega(d))^{1/q} \times \frac{(\Gamma(q+1))^{1/q}}{(q(d-\alpha)-d)^{1+1/q}} \geq$$

$$D_{\alpha,d} \cdot (d/\alpha - p)^{1+1/q} \cdot (p-1)^{-1-1/q};$$

recall that  $q = q(p)$ .

**Second example.** We put:

$$g_0(x) = |x|^{-\alpha} I(|x| < 1),$$

and find:

$$|g_0|_p = \frac{\omega^{1/p}(d) \alpha^{-1/p}}{(d/\alpha - p)^{1/p}} \leq a(\alpha, d) \cdot (d/\alpha - p)^{-\alpha/d}, \quad p \in (1, d/\alpha);$$

$$v_0(x) := I_\alpha g_0(x) \geq 3^{-1} \cdot 4\pi \cdot \omega(d-1) \cdot 2^{-d} \cdot |\log|x|| \cdot I(|x| < 1);$$

$$|v_0|_q \geq \omega^{1/q} \cdot d^{-1-1/q} \cdot \Gamma^{1/q}(q+1) \geq A(\alpha, d) \cdot (d/\alpha - p)^{-1}, \quad p \in (1, d/\alpha).$$

**Third example. Summing.**

We define:

$$h(x) = f_0(x) + g_0(x).$$

It follows from the triangle inequality:

$$|h|_p \leq |f_0|_p + |g_0|_p \leq a(\alpha, d) (d/\alpha - p)^{-\alpha/d} + n(\alpha, d) (p-1)^{-1/p}.$$

Further, as long as both the functions  $u_0$  and  $v_0$  are positive, we obtain for the values  $p$  from the interval  $p \in (1, d/\alpha)$ :

$$|I_\alpha h|_q \geq 0.5(|u_0|_q + |v_0|_q) \geq A(\alpha, d) (d/\alpha - p)^{-1} + D(\alpha, d) (d/\alpha - p)^{1+1/q} (p-1)^{-1-1/q}.$$

We get after dividing:

$$2 \frac{|I_\alpha h|_q \cdot [(p-1)(d/\alpha - p)]^{1-\alpha/d}}{|h|_p} \geq \frac{A(p-1)^{1+1/q} + D(d/\alpha - p)^{2+1/q}}{a(p-1)^{1/p} + n(d/\alpha - p)^{\alpha/d}},$$

i.e.

$$\frac{|I_\alpha h|_q \cdot [(p-1)(d/\alpha - p)]^{1-\alpha/d}}{|h|_p} \geq F_{\alpha,d}(p),$$

following

$$\frac{|I_\alpha h|_q \cdot [(p-1)(d/\alpha - p)]^{1-\alpha/d}}{|h|_p} \geq R(\alpha, d), \quad (11)$$

which is equivalent to the assertion of theorem 2.

### 3. THE CASE OF A BOUNDED DOMAIN

We consider in this section the *truncated* Riesz's operator

$$u^{(G)} = u^{(G)}(x) = I_\alpha^{(G)} f(x) = \int_G \frac{f(x-y) dy}{|y|^{d-\alpha}}, \quad (12)$$

where  $G$  is open bounded domain in  $R^d$  containing the origin and such that

$$0 < \inf_{x \in \partial G} |x| \leq \sup_{x \in \partial G} |x| < \infty,$$

$\partial G$  denotes boundary of the set  $G$ .

It is known (see, e.g. [11], p.90), that if  $f \cdot \max(1, \log |f|) \in L_1(G)$ , then  $I_\alpha^{(G)} f \in L_q(G)$ .

We can and will assume further without loss of generality that the set  $G$  is unit ball in the space  $R^d$ :

$$G = \{x, x \in R^d, |x| \leq 1\}.$$

Let us denote as ordinary  $p' = p/(p-1)$ ,  $p \in (1, \infty)$  and introduce the correspondent bilinear truncated Riesz's functional:

$$B^{(G)}(f, g) = \int_{R^d} g(x) dx \left\{ \int_G \frac{f(y) dy}{|x-y|^{d-\alpha}} \right\} = (I_\alpha^{(G)} f, g). \quad (14)$$

Let us denote

$$Z(p) = \frac{[\omega(d)/(d-\alpha)]^{1/p}}{(d/(d-\alpha) - p)^{1/p}},$$

where the variable  $p$  is the following function on the variables  $r$  and  $s$ :

$$p = p(r, s) = \frac{r'/s'}{r'/s'},$$

and the variable  $r, s$  are such that

$$r, s \geq 1, 1 \leq 1/r + 1/s < 1 + \alpha/d.$$

#### Theorem 3.

$$|B^{(G)}(f, g)| \leq Z(p(r, s)) |f|_r |g|_s. \quad (15)$$

**Proof** is very simple by means of the *classical* Young's inequality (the *weak* Young's inequality gives at the same result).

Namely, we obtain by the direct computation for the values  $p$  from the interval  $p \in [1, d/(d - \alpha))$ :

$$\| |x|^{\alpha-d} \cdot I(x \in G) \|_p = Z(p).$$

Note that in contradiction to the unbounded case  $G = R^d$  the values  $(r, s)$  have two degrees of freedom.  $\square$

#### 4. CONCLUDING REMARKS

**A.** We consider in this subsection some generalization of the Riesz's potential (non-truncated) operator of a view

$$I_{\alpha,\beta} f(x) = \int_{R^d} \frac{f(y) |\log|x-y||^\beta dy}{|x-y|^{d-\alpha}},$$

$\alpha = \text{const} \in (0, d)$ ,  $\beta = \text{const} > 0$ , or equally

$$I_{\alpha,\beta} f(x) = \int_{R^d} \frac{f(y) [1 + |\log|x-y||]^\beta dy}{|x-y|^{d-\alpha}},$$

or more generally

$$I_{\alpha,\beta}^{(Q)} f(x) = \int_{R^d} \frac{f(y) |\log|x-y||^\beta Q(|\log|x-y||) dy}{|x-y|^{d-\alpha}}, \quad (16)$$

where  $\alpha = \text{const} \in (0, d)$ ,  $\beta = \text{const} > 0$ , and  $Q(z)$  is a *slowly varying* as  $z \rightarrow \infty$  continuous positive function:

$$\forall \lambda > 0 \Rightarrow \lim_{z \rightarrow \infty} Q(\lambda z)/Q(z) = 1.$$

It is proved in the article [14] by means of considerations used in the proof of theorem 1 that

$$|I_{\alpha,\beta}^{(Q)} f|_q \leq \frac{C |f|_p}{[(p-1) \cdot (d/\alpha - p)]^{1+\beta-\alpha/d}} \cdot Q(q(d-\alpha) - d)^{-1} Q(q), \quad (17)$$

and that the last inequality (17) is asymptotically as  $p \rightarrow 1+0$  and  $p \rightarrow d/\alpha - 0$  exact, e.g. for the function  $h(\cdot)$ .

Here as in the sections 1,2  $p \in (1/d/\alpha)$  and  $q = q(p)$ .

Therefore, we have for the corresponding *generalized bilinear Riesz's functional*

$$B_{\alpha,d}^{(Q)}(f, g) := (g, I_{\alpha,\beta}^{(Q)} f),$$

$$V_{\alpha,d}^{(Q)}(r, s) := \sup_{f \in L_r, f \neq 0} \sup_{g \in L_s, g \neq 0} \frac{B_{\alpha,d}^{(Q)}(f, g)}{|f|_r |g|_s}$$

the following bilateral estimation:

**Theorem 4.**

$$\frac{C_1(d, Q(\cdot)) \alpha^{-1-\beta+\alpha/d} Q(1/\alpha)}{[(r-1)(s-1)]^{1+\beta-\alpha/d}} \leq V_{\alpha,d}^{(Q)}(r, s) \leq \frac{C_2(d, Q(\cdot)) \alpha^{-1-\beta+\alpha/d} Q(1/\alpha)}{[(r-1)(s-1)]^{1+\beta-\alpha/d}}, \quad (18)$$

where as in (2)

$$r, s > 1, \frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{d}.$$

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